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Closed Economy Analysis of the Dynamics of Deficits and Debt In a Unitary and Federal State

by

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Abstract

This study re-examines the government debt-to-GDP stability conditions derived from the single equation first difference analysis of the dynamics of deficits and debt. The theoretical models discussed in this paper reveal potential bias in the stability conditions from at least two sources. First, the assumption of constant interest rates, real growth and inflation is shown to be valid only in a Ricardian world in which government bonds are not perceived to be part of net wealth. Second, the assumption of a unitary state, by abstracting from the dynamic interactions between different levels of government, is demonstrated to bias the results even if Ricardian equivalence is imposed.

The novelty of this paper is to introduce simulation exercises in order to assess the impact of relaxing these assumptions on the level of the steady-state government debt-to-GDP ratio. The impact is shown to be a function of the model parameters and the degree to which agents perceive government bonds to be part of net wealth. For example, for a conventional set of parameters in the non-Ricardian case, a one percentage point decrease in taxes in the unitary state model results in a 4 percentage point higher debt-to-GDP ratio when compared to the single equation first difference framework. The same shock in the federal state model, in the Ricardian case, results also in a 4 percentage point higher debt-to-GDP ratio at the federal level when compared to the simpler approach.

Résumé

Cette étude examine la validité de la condition de stabilité du rapport dette PIB provenant de l'analyse en temps discret d'une équation représentant la dynamique des déficits et de la dette. Les modèles considérés dans cette étude révèlent au moins deux biais possibles à cette approche. Premièrement, traiter les taux d'intérêt, d'inflation et de de croissance économique comme des paramètres constants requiert l'hypothèse d'un monde Ricardien ou les bons du gouvernement ne sont pas perçu comme de la richesse privée. Deuxièmement, l'hypothèse d'un état unitaire, en faisant abstraction des interactions existants entre les différents niveau de gouvernements, impose un biais même si on maintient l'hypothèse d'un monde Ricardien.

L'aspect novateur de cette étude est d'introduire une analyse de simulation de manière a quantifier l'impact de relâcher ces hypothèses sur la valeur à l'équilibre stationnaire du rapport dette PIB. Ces simulations démontrent que cet impact est une fonction des parametres du modèle et de la proportion dans laquelle les agents perçoivent les bons du gouvernement comme faisant partie de leur richesse. Par exemple, dans le cas non-Ricardien et pour des valeurs conservatrices des paramètres, une diminution d'un point de pourcentage des taxes résulte en un rapport dette PIB plus élevé de 4 points de pourcentage par rapport à l'approche impliquant une équation dynamique unique en temps discret. Le même choc effectué dans le modèle d'une économie fédérale implique également une augmentation de 4 points de pourcentage du rapport dette PIB par rapport au résultat obtenue dans l'approche plus simple.

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1. Introduction

Since the fall of 1984, the government has pursued as a fiscal objective a policy of halting the growth of the debt-to GDP ratio. This has prompted a substantial amount of research in the Department and in the academic journals on the economic conditions required to achieve this objective. A substantial portion of this analysis has been conducted in the framework of a single dynamic equation relating growth in the debt-to-GDP ratio to the primary balance, growth in output and interest rates.[1] Despite the simplicity of this type of analysis -- in that it assumes constant interest rates and inflation, and the existence of only one level of government -- it has been very useful in that intuitively appealing stability conditions with important policy implications can be derived. Nevertheless, given the strong assumptions required to conduct this type of analysis, economic theory suggests that the results could be exposed to serious bias.

This paper examines the validity of the single equation first difference approach in determining the conditions for stability and steady-state values of the debt-to-GDP ratio. With this purpose in mind, the federal government budget constraint is incorporated into a closed IS-LM model of the economy, first excluding and second including a provincial government sector in order to:

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1. See Darby (1984), Fortin (1984), Albert (1986), Young (1987).

1. determine the macroeconomic context in which the assumptions of parameter constancy and a unitary state are appropriate; and,

2. identify through simulation exercises, the implied bias from a single equation-first difference framework in the case where the above assumptions do not hold.

The theoretical models discussed in this paper reveal bias in the single equation-first difference framework from at least two sources. First, assuming parameter constancy requires a Ricardian world in which government bonds are not perceived to be part of net wealth. Failure of this assumption implies that the stability conditions may actually be more difficult to achieve than previously considered. In fact, when the assumption of parameter constancy with respect to interest rates and inflation is relaxed, the stability conditions must take into account two additional factors: the impact of a change in interest rates on interest paid on the government debt and, the change in inflation tax from a change in inflation. Second, the assumption of a unitary state is sufficient to impose serious bias on the results regardless of the assumption of Ricardian equivalence. In fact, examination of a model including a provincial sector that bond finances its deficit reveals that the stability conditions are more difficult to achieve and debt-to-GDP steady-state values are higher even if interest rates and inflation are held constant.

The remainder of the paper is divided into four sections. The following section briefly addresses theoretical aspects of deficit and debt dynamics in the context of papers by Young (1987), Scarth (1987), and Kneebone (1988). Analysis of two closed IS-LM models of the economy is conducted in section 3. Using simulation experiments, section 4 provides some empirical perspective on the direction and magnitude of the bias involved in the constant parameter and unitary state assumptions of the finite first difference framework, and section 5 draws some conclusions. An open economy version of the analysis is currently underway and will be available in a subsequent paper.

2. Theoretical Considerations

Single equation first difference analysis of deficit and debt dynamics generally begins with a definition of a simple government budget constraint:[2]

$$\text{(i)} \quad \frac{\dot{B}(t)}{P(t)} = G(t) - T(t)Y(t) + r(t)[1-T(t)]\frac{B(t)}{P(t)}$$

where B(t) is the stock of net debt, G(t) denotes real budgetary expenditures less public debt charges, Y(t) is real gross domestic product (GDP), T(t) is the proportional tax rate, P(t) represents the price level, and r(t) is the interest rate on the stock of net debt.

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2. For simplicity the theoretical analysis is conducted in a continuous time (rather than discrete time) framework.

Time derivatives are denoted with a dot. Manipulation of equation (i) reveals that the change in government debt, is the sum of the primary balance, after-tax interest costs of servicing the debt, and the real after-tax less growth rate effect of lagged debt, all expressed relative to GDP:

$$\text{(ii)} \quad \dot{b}(t) = pd(t) + r(t)b(t) + [r(t) - y(t)]\, b(t-1)$$

Where pd represents the primary deficit as a percentage of GDP, b(t) is the real debt as a percentage of GDP, and y(t) is the contemporaneous rate of growth in GDP. Assuming interest rates, inflation and growth are constant over time, the condition for convergence of the debt-to-GDP to steady-state is given by:

$$\text{(iii)} \quad n > r(1-T)-p$$

where p is the rate of price inflation and n is contemporaneous growth in GDP. Given that equation (iii) allows only two possible cases, the implications for fiscal policy are clear. Specifically, if equation (iii) is satisfied, the debt-to GDP ratio is stable even if the government chooses to maintain permanent primary deficits. If equation (iii) is <u>not</u> satisfied, the debt-to-GDP ratio will explode unless the government reacts and sustains sufficiently large primary surpluses. It is important to note that whether or not an equilibrium exists where (iii) is satisfied is the subject of debate.[3] However,

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3. For example, see Barro (1976), Feldstein (1976), Carmichael (1982), Burbidge (1983).

whether or not (iii) can be ever satisfied does not invalidate the fact that it is a necessary condition for stability in this model. Discussion and simulations in this paper are limited to the case where equation (iii) is satisfied.

Given (iii) is satisfied, an equilibrium debt-to-GDP ratio exists and is as follows:

$$(iv) \quad b(e) = \frac{pd}{n-[r(1-T)-p]}$$

where e denotes equilibrium values. For example, given a primary deficit equal to 2 per cent of GDP, interest rates equal to 8 per cent, inflation equal to 4 per cent, growth equal to 6 per cent, and a 17 per cent tax rate, the equilibrium debt-to-GDP ratio is approximately 60 per cent.

It is at this point in single equation first difference analysis of deficit and debt dynamics that "what if?" scenarios are simulated under the assumption that the standard stability condition is sufficient for convergence. For example, Young (1987) determines a primary surplus required to stabilize the debt-to-GDP ratio should real interest rates exceed growth. As well, given projected values for the economic variables involved, a time path for the components of the debt-to-GDP ratio is constructed.

While these exercises can be informative, the implications of using such a simple approach must be considered. Specifically, because the model assumes parameter constancy, "second round" effects associated

with any shock are not taken into account. If the model were extended to include a provincial government sector, more stringent conditions must be satisfied to achieve debt-to-GDP stability.[4] Moreover, examination of an open economy version of the model requires an additional stability condition.[5]

Scarth (1987) addresses the debt stability question through development of a closed and then open IS-LM model of the economy. Starting with a simple government budget constraint, and then adding an IS-LM system, Scarth determines that the assumption of Ricardian equivalence is sufficient to produce a dichotomy in the system. Specifically, if agents perceive issuance of government bonds to imply a future tax liability, and the government bond finances its deficit then the perceived private sector net wealth should be unaffected by the larger stock of government bonds. This severs the link between inflation, interest rates and government debt. As a result, in the closed economy case, assuming parameter constancy in single equation first difference analysis of the government budget constraint has no implied bias. That is, there is no change in stability conditions and steady-state values when a more complete model of the closed economy is specified.

While the assumption of parameter constancy is justified in a Ricardian world, the implications of bonds being perceived as part of net wealth must be considered. Working with an altered version of a

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4. Kneebone (1988).
5. Scarth (1987)

simple IS-LM model proposed by Kneebone (1988), analysis reveals that, in a non-Ricardian world, a link exists between bonds, interest rates and inflation and the assumption of parameter constancy is invalid. In fact, the standard stability condition, given by equation (2) is neither a necessary nor sufficient condition for convergence to a constant debt-to-GDP ratio in this case.

In Kneebone's (1988) examination of the macroeconomic effects of government deficits and debt, he points out that the implicit assumption of a unitary state, which is prevalent in literature, is sufficient to impose serious bias on results obtained regardless of the assumption of Ricardian equivalence. In particular, he finds that when a second government budget constraint is added to incorporate the provincial level of government, and provincial governments are assumed to bond finance their deficits, stability and steady state values of the federal debt-to-GDP ratio are affected <u>even if Ricardian equivalence is imposed</u>.

While the results of Kneebone (1988) and Scarth (1987) are important in that they reveal some bias in the standard results, the direction and the magnitude of this bias is indeterminate. This paper uses simulation techniques to identify the direction and magnitude of comparative dynamic results that would allow the bias to be quantified. The following section analyses two versions of Kneebone's model for the purpose of reviewing the theoretical framework necessary in determining the necessary and sufficient stability conditions under varying assumptions and in setting the stage for simulations in Section 4.

3. Closed Economy Analysis

Derivation of debt-to-output stability conditions involves specification of an IS-LM system of equations, and examination of the relationship between bonds and other variables in the model. As previously discussed, this relationship depends primarily on two conditions. First, the degree to which agents perceive government bonds to be part of net wealth. Examination of a simple IS-LM model of an economy with one level of government will illustrate this point. Second, the extent that provincial decisions on how to finance their debt affects the federal government's budget constraint. Formalization of this point involves extension of the previous IS-LM model of a unitary state to a federal economy.

(i) Unitary State

The following IS-LM system describes a closed economy with flexible prices, and rational economic agents that have perfect foresight. The model includes only one level of government and is deterministic. For simplicity, the single level of government is labelled federal. Stability conditions are generated from manipulation of the following seven equations:

$$(1) \qquad Y = C(Y^d, r - \Pi, W) + G^f$$

$$[C = C_1 Y^d + C_2'(r - \Pi) + C_3 W]$$

$$(2) \qquad W = \frac{M}{P} + \beta \frac{B^f}{P}$$

$$(3) \qquad Y^d = (1 - t^{fy})Y + r(1 - t^{fr})\frac{B^f}{P}$$

$$- \Pi W - (1 - \beta)\frac{\dot{B}^f}{P} + TR^f$$

$$(4) \qquad \frac{M}{P} = L(Y, r, W) = L_1 Y + L_2' r + L_3 W$$

$$(5) \qquad p = \theta(Y - \bar{Y}) + \Pi$$

$$(6) \qquad \Pi = p$$

$$(7) \qquad \frac{1}{P}(\dot{M} + \dot{B}^f) = G^f + TR^f - t^{fy}Y + r(1 - t^{fr})\frac{B^f}{P}$$

where:

Y - real output

$\bar{Y}$ - real capacity output

r - nominal interest rate

Π - expected rate of inflation

P - price level

$p = \frac{\dot{P}}{P}$ - rate of inflation

B^f - nominal value of federal government bonds outstanding

M - nominal money supply

Y^d - real private disposable income

t^{fy} - average tax rate at the federal level

t^{fr} - marginal tax rate on interest payments at the federal level

β - fraction of government bonds which private agents perceive as net wealth ($0 \leq \beta \leq 1$)

W - private agents subjective value of real net wealth

θ - rate at which inflation rises due to excess demand in the product market ($\theta > 0$)

G^f - real federal government expenditures on goods and services

TR^f - real federal government transfers

C - real private expenditure

and time derivatives are denoted with a dot.

Equation (1), the GNP identity, assumes consumption is a linear function of real disposable income, real interest rates and real wealth, and that there is no capital formation. Unless there was 100 per cent depreciation, the inclusion of capital formation would imply a different definition of wealth than (2). Denoting partial derivatives with subscripts, the following relationships are assumed:[6]

$$0 < C_{yd} < 1, \quad C_{r-\Pi} < 0, \quad C_{w} > 0$$

Equation (2) defines net wealth to explicitly incorporate the degree to which agents associate bond issues with future tax liabilities. For example, if $\beta = 0$, Ricardian equivalence is imposed so that real bonds do not make any contribution to net wealth. A Keynesian assumes the $\beta > 0$ scenario where bond financed deficits have some effect on aggregate demand.

Real disposable income, the maximum rate at which economic agents can consume while expecting that their real wealth is unchanged, is derived by manipulation of the wealth equation, government budget constraint, and GNP identity. Specifically, taking the time derivative of (2):

$$(8) \qquad \dot{W} = \frac{1}{P}[\dot{M} + \dot{B}^{f}] - \Pi W - (1 - \beta)\frac{\dot{B}^{f}}{P}$$

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6. Ott, Ott and Yoo (1975, pp. 84-87) show that in a constant growth model the long-run MPC is less than unity.

equating (8) to zero, and substituting in the government budget constraint and the GNP identity results in (3). This is analogous to Kneebone's representation of disposable income as the second last term includes the savings perceived as being necessary to finance the implied future tax liability from issuance of government bonds.

Equation (4) requires real money supply equal real money demand. Real output, nominal interest rates and real net wealth influence demand for real money balances and the following relationships are assumed:

$$L_y > 0, \quad L_r < 0, \quad 0 < L_w < 1$$

Equation (5) denotes the expectations augmented Phillips curve with the "accelerationist" view incorporated that, after consideration of equation (6), implies real capacity output is maintained. Kneebone suggests that while the assumption of perfect foresight may not be appropriate in the short run, it is appropriate in a long-run equilibrium context.[7]

The government budget constraint, equation (7), requires that uses of funds equal sources of funds at all times.

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7. Kneebone, page 6.

Equations (1) through (7) describe an economy with zero economic growth. Since the rate of growth is important to the analysis of debt dynamics, Kneebone's assumption of a constant exogenously determined rate of real growth is adopted. To include this assumption, the model is divided through by Y and further simplified by substituting equations 2, 5 and 6 into the rest of the model.

The resulting four equations that describe a closed economy with constant exogenous economic growth and rational economic agents are as follows:

$$(9) \qquad 1 = C_1 y^d + C_2(r - p) + C_3[m + \beta b^f] + g^f$$

$$(10) \qquad y^d = (1 - t^{fy}) + r(1 - t^{fr})b^f - p(m + b^f) - (1 - \beta)[\dot{b}^f + nb^f]$$

$$(11) \qquad m = L_1 + L_2 r + L_3[m + \beta b^f]$$

$$(12) \qquad \dot{m} + \dot{b}^f = g^f + tr^f - t^{fy} + r(1 - t^{fr})b^f - (m + b^f)(n + p)$$

where:

$$y^d = \frac{Y^d}{\overline{Y}} \quad ; \quad g^f = \frac{G^f}{\overline{Y}} \quad ; \quad C_2 = \frac{C'_2}{\overline{Y}} \quad ; \quad B^f = \frac{b^f}{\overline{PY}} \quad ;$$

$$\frac{M}{\overline{PY}} = m \quad ; \quad L_2 = \frac{L'_2}{\overline{Y}} \quad ; \quad n = \frac{\dot{Y}}{\overline{Y}} \quad ; \quad TR^f = \frac{tr^f}{\overline{Y}}$$

In this form of the model y^d, p, r and b^f are defined as endogenous variables with m, t^{fy}, t^{fr}, tr^f and g^f assumed exogenous. The government is assumed to follow a monetary rule that allows real money balances to grow at a rate that ensures the real money balance-

to-real output ratio remains constant ($\dot{m}=0$). Kneebone (1989) defines an alternative money growth rule where the rate of growth in the nominal money supply is set equal to a constant ($\dot{M}/M = \alpha$). In this case, Kneebone finds an unstable root that prevents the model from evolving toward a new steady-state. In order to achieve a steady-state, he assumes that agents cause inflation to "jump" instantaneously to its new equilibrium level. This ensures that $\dot{m}=0$ and, with the unstable root eliminated, that a steady-state is possible. Since the results of Kneebone's analysis are the same under both monetary rules (and given the assumption about inflation as a "jump" variable), the assumption of $\dot{m}=0$ will be retained. With $\dot{m} = 0$ equation (12) becomes:

$$\dot{b}^f = g^f + tr^f - t^{fy} + r(1 - t^{fr})b^f - (m + b^f)(n + p) \tag{13}$$

Substituting (13) into (10) and simplifying results in:

$$y^d = 1 - (g^f + tr^f) + (1 - \beta)\, nm + \beta[g^f + t^{fr} - t^{fy} + r(1 - t^{fr})b^f - p\,(m + b^f)] \tag{14}$$

The model now consists of three static equations (9), (11) and (14), and one asset accumulation identity (13). In order to determine the effects of changes in government bonds on the government

debt-to-output ratio, the relationship between b^f, r, and p must be made explicit. The first step in solving the static equations for r and p as functions of b^f and exogenous variables is to solve (11) for r:

$$(15) \qquad r = \frac{1}{L_2} m - \frac{L_3}{L_2}(m + \beta b^f) - \frac{L_1}{L_2}$$

Since wealth and income are determined elsewhere in the model, the link between r and b^f can be found immediately by differentiating (15) with respect to b^f:

$$(16) \qquad \frac{\partial r}{\partial b^f} = - \frac{\beta L_3}{L_2} \gtrless 0$$

This reveals a positive relationship between interest rates and bonds. Intuitively, a ceteris paribus increase in government bonds causes both a rightward shift in the IS curve and a leftward shift in the LM curve. These movements are induced by a wealth effect on both consumption and demand for real money balances. The next step involves solving (9) for p, substituting (15) for r, and (13) for y^d to obtain:

$$(17) \qquad p = \frac{K + \beta b^f Z}{L_2[C_2 + C_1\beta(m + b^f)]}$$

where:

$$(18) \qquad K = L_2C_1[1 - (g^f + tr^f) - (1 - \beta)\, nm + \beta(g^f + tr^f - t^{fy})] - C_2m - L_3C_2m + L_2C_3m + L_2g^f - L_2 - L_1C_2$$

and

$$(19) \qquad Z = (L_2C_1(1 - t^{fr})\, r) + (C_3L_2 - L_3C_2)$$

Differentiating (17) with respect to b^f results in:

$$(20) \qquad \frac{\partial p}{\partial b^f} = \beta\, \frac{L_2\{C_1[(1 - t^{fr})\, r - p] + C_3\} - L_3[C_2 + \beta C_1(1 - t^{fr})b^f]}{L_2[C_2 + C_1\beta(m + b^f)]} \lesseqgtr 0$$

implying an equivocal relationship between inflation and bonds. The ambiguity stems from the fact that a ceteris paribus increase in government bonds, in the absence of Ricardian equivalence, has an indeterminate affect on aggregate demand. Following Rau (1985), the effects of a change in government bonds on aggregate demand can be decomposed into 3 components:

(i) An increase in bonds implies an increase in interest payments to private agents and will have a positive effect on consumption (transfer effect) through the impact on disposable income.

(ii) To the extent that government bonds are perceived as net wealth, an increase in bonds will have a positive impact on consumption (wealth effect).

(iii) Given that government bonds are part of perceived net wealth, an increase in bonds will stimulate demand for real balances (money-market effect).

Following a ceteris paribus increase in government bonds, (i) and (ii) imply a rightward shift in the IS curve and (iii) implies a leftward shift in the LM curve. The overall effect on inflation is uncertain as it is ambiguous whether the situation is one of excess demand or excess supply.

It is clear at this point that if Ricardian equivalence was imposed ($\beta = 0$), both (16) and (20) would equal zero and any influence of b on r and p would be nullified.

Referring to the government budget constraint defined by (13), and noting that it includes endogenous variables b^f, r and p, stability conditions can be derived by differentiating (13) with respect to b^f. Recalling results (16) and (20), convergence to a constant federal debt-to output ratio requires:

$$(21) \qquad \frac{\partial \dot{b}^f}{\partial b^f} = -[n - (r(1 - t^{fr}) - p)] + [b^f(1 - t^{fr})] \frac{\partial r}{\partial b^f} - (m + b^f) \frac{\partial p}{\partial b^f} < 0$$

If Ricardian equivalence is imposed ($\beta = 0$), the necessary and sufficient conditions for convergence are:

$$(22) \qquad n > r(1 - t^{fr}) - p$$

given g^f, t^{fr}, t^{fy}, m and n are exogenous. This is not a surprising result and is consistent with Scarth (1987) and Young (1987).

However, if $\beta > 0$ then (22) is no longer a necessary condition for convergence to a stable debt-to-output ratio. In this case, the necessary and sufficient condition for convergence involves two additional components: the impact of a change in debt on interest paid on government debt, and the change in inflation tax from a change in debt. Intuitively, if agents perceive government bonds to be a part of their net wealth, both interest on the public debt and the inflation tax become sensitive to debt policy. For example, if an increase in government bonds induces an increase in interest rates and a decrease in inflation, growth exceeding real after tax interest rates may not be sufficient to stabilize the debt-to-GDP ratio. Alternatively, if an increase in government bonds increases inflation, growth plus inflation tax may be enough to stabilize the debt-to-GDP ratio.

(ii) Federal Economy

Extension of the analysis to include the effects of provincial government budget decisions on macroeconomic stability simply involves redefinition of the previous model. It is important to note that this analysis is not equivalent to the study of an IS-LM system with one budget constraint at the total government level.[8] Following Kneebone (1988), the provincial government sector's expenditures, taxation and budget constraint are incorporated into the model as follows:

$$(23) \qquad Y = C(Y^d, r - \Pi, W) + G^f + G^p$$

$$[C = C_1 Y^d + C_2'(r - \Pi) + C_3 W]$$

$$(24) \qquad W = \frac{M}{P} + \beta\left[\frac{B^f}{P} + \frac{B^p}{P}\right]$$

$$(25) \qquad Y^d = (1 - t^{fy} - t^{py})Y + r(1 - t^{fr} - t^{pr})\left(\frac{B^f}{P} + \frac{B^p}{P}\right)$$

$$- \Pi W - (1 - \beta)\left(\frac{\dot{B}^f}{P} + \frac{\dot{B}^p}{P}\right) + TR^f + TR^p$$

$$(26) \qquad \frac{M}{P} = L(Y, r, W) = L_1 Y + L_2' r + L_3 W$$

$$(27) \qquad p = \theta(Y - \bar{Y}) + \Pi$$

$$(28) \qquad \Pi = p$$

$$(29) \qquad \frac{1}{P}(\dot{M} + \dot{B}^f) = G^f + TR^f - t^{fy}Y + r(1 - t^{fr})\frac{B^f}{P} - t^{fr}\frac{r}{r}\frac{B^p}{P}$$

$$(30) \qquad \frac{1}{P}\dot{B}^p = G^p + TR^p - t^{py}Y + r(1 - t^{pr})\frac{B^p}{P} - t^{pr}\frac{r}{r}\frac{B^f}{P}$$

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8. For further discussion on this point see Kneebone (1988, pp.17-18).

where the superscripts f and p refer to federal and provincial government sectors respectively. Otherwise the notation is as previously outlined.

There are two implicit assumptions of note. First, in order to allow an exchange of federal for provincial government bonds in the absence of a change in bond prices, for simplicity it is assumed that they are regarded as perfect substitutes. Second, it is assumed that all provincial governments finance changes to their budget positions in the same manner, so that equation (30) represents the aggregate non-federal budget constraint.

Derivation of stability conditions for the debt-to-GDP ratio involves determination of the relationship between bonds, interest rates and inflation, and specification of financing options of both levels of government. Assuming the federal government follows a money growth rule where $\dot{m} = 0$, equations (23) to (30) can be manipulated in a fashion similar to that of the previous section in order to reveal the following relationships:

$$(31) \qquad \frac{\partial r}{\partial b} \equiv \frac{\partial r}{\partial b^f} = \frac{\partial r}{\partial b^p} = -\beta \frac{L_3}{L_2} \gtrless 0$$

and,

$$(32) \qquad \frac{\partial p}{\partial b} \equiv \frac{\partial p}{\partial b^f} = \frac{\partial p}{\partial b^p}$$

$$= \beta \frac{L_2[C_3 + C_1(r(1-t^{fr} - t^{pr}) - p)] - L_3[C_2 + \beta C_1(1 - t^{fr} - t^{pr})(b^f + b^p)]}{L_2[C_2 + \beta C_1(m + b^f + b^p)]} \gtrless 0$$

Equations (31) and (32) define relationships similar to those in the previous section. Again the link between inflation and bonds, although slightly altered, is ambiguous in sign if $\beta > 0$. This reflects the effect of b^f and b^p on wealth, and hence consumption and money demand.

It is necessary at this point to define the financing strategy of provincial governments. Since this sector can only tax finance or bond finance shocks to their budget positions, both cases will be considered.

(a) Provincial Tax Financing

Assuming that all provincial governments tax finance ($\dot{b}^p = 0$) and the federal government follows a monetary growth rule ($\dot{m} = 0$), equation (30) becomes static and (29) becomes:

$$(33) \qquad \frac{\dot{B}}{P}^f = G^f + TR^f - t^{fy}Y + r(1 - t^{fr})\frac{B}{P}^f - t^{fr}\frac{B}{P}^p$$

Differentiating (33) with respect to b^f reveals that convergence to a constant federal debt-to-GDP ratio requires:

$$(34) \qquad \frac{\partial \dot{b}^f}{\partial b^f} = -[n - (r(1 - t^{fr}) - p)] + [b^f (1 - t^{fr}) - t^{fr} b^p] \frac{\partial r}{\partial b^f}$$

$$- (m + b^f) \frac{\partial p}{\partial b^f} < 0$$

In a Ricardian world ($\beta=0$), the necessary and sufficient condition becomes:

$$(35) \qquad n > r\,(1 - t^{fr}) - p$$

which is identical to the results of equation (22) in the previous section. This result is intuitive because, in the absence of federal or provincial government bonds being considered a part of net wealth, the growth in the federal tax base alone must be sufficient to stabilize the federal debt-to-GDP ratio.

However, if $\beta>0$ the condition for stabilization of the debt-to-GDP ratio, equation (34), differs from the previous section in two respects. First, an increase in interest rates following an increase in government bonds impacts less on the stability conditions than in the unitary state case. This is because the effect of an increase in interest rates on interest paid on the federal debt is reduced by federal tax revenue received on interest income from provincial bonds. Second, as revealed by equation (32), the effect of a change in federal government bonds on inflation takes into account the tax rate and the debt-to-GDP ratio of the provincial government sector.

(b) Provincial Bond Financing

Assuming that all provincial governments bond finance ($\dot{b}^{p} \neq 0$) and the federal government continues to follow a monetary rule ($\dot{m} = 0$),

two different concepts of equilibrium can be defined. First, equilibrium can require that only the sum of the change in provincial and federal budget deficits equal zero (ie. $\dot{b}^T \equiv \dot{b}^f + \dot{b}^p = 0$). In this case, adding equations (30) and (33) and differentiating with respect to b^T reveals that convergence to this type of equilibrium requires:

$$(36) \qquad \frac{\partial \dot{b}^T}{\partial b^T} = -[n - (r(1 - t^{fr} - t^{pr}) - p)] - b^T[1 - t^{fr} - t^{pr}] \frac{\partial r}{\partial b} + (m + b^T) \frac{\partial p}{\partial b} < 0$$

However, equation (36) defines an equilibrium which may be unsustainable. Because only the sum of $\dot{b}^f$ and $\dot{b}^p$ must equal zero, this represents only a quasi-equilibrium in that one level of government could be continually issuing new bonds while the other is retiring bonds. To the extent that a "crisis of confidence" may result, this equilibrium may not be sustainable and adjustment to a full equilibrium must occur.

Full equilibrium, requires that both levels of government maintain a constant debt-to-GDP ratio. Given this second definition of equilibrium, stability conditions are derived by considering equations (30) and (33) separately. Expanding the following determinant,

$$(37) \qquad \begin{vmatrix} \frac{\partial \dot{b}^f}{\partial b^f} - \lambda & \frac{\partial \dot{b}^f}{\partial b^p} \\ \frac{\partial \dot{b}^p}{\partial b^f} & \frac{\partial \dot{b}^p}{\partial b^p} - \lambda \end{vmatrix} = 0$$

where λ is an eigenvalue of the characteristic equation and applying the Routh Theorem, the two necessary and sufficient conditions for convergence are as follows[9]:

$$(38) \qquad [n - (r - p)]\, A > 0$$

$$(39) \qquad [n - (r - p)] + A > 0$$

where,

$$A \equiv [n - (r(1 - t^{fr} - t^{pr}) - p)] - b^{T}(1 - t^{fr} - t^{pr}) \frac{\partial r}{\partial b} + (m + b^{T}) \frac{\partial p}{\partial b}$$

and

$$b^{T} = b^{f} + b^{p}$$

It can be shown that satisfaction of equations (38) and (39) requires:

$$(40) \qquad n > r - p$$

and,

$$(41) \qquad n > [r\ (1 - t^{fr} - t^{pr}) - p] + b^{T}(1 - t^{fr} - t^{pr}) \frac{\partial r}{\partial b} - (m + b^{T}) \frac{\partial p}{\partial b}$$

If $\beta = 0$, then (40) is the necessary and sufficient condition for convergence to a stable debt-to-GDP ratio. This is an important result since (40) is a considerably more stringent condition than the unitary state condition with Ricardian Equivalence (equation (22)) and is necessary regardless of the value of β. If $\beta > 0$ then both (40) and (41) are required for convergence.

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9. Kneebone (1988), p. 51.

In effect, application of the standard stability conditions (equations (22) and (35)) significantly overstates the ability of the federal economy to achieve convergence when the federal government follows a monetary growth rule, irrespective of the value of β. Intuitively, if a unitary government operates in a deficit position, after-tax interest payments must grow at a rate slower than the economy in order to achieve a stable debt-to-GDP ratio. However, if a provincial government exists, the fact that the federal government is issuing bonds means the provinces enjoy smaller deficits (through the $t^{pr}rb^{f}$ term) and therefore issue fewer bonds. The federal government is even worse off in this event because they receive less tax revenue on income from provincial bonds ($t^{fr}rb^{p}$ smaller).

Theory developed so far indicates that the robustness of the results of the single equation first difference analysis depends on at least two very restrictive assumptions. Specifically, the assumption of parameter constancy is only valid in a Ricardian world. In addition, stability conditions derived under this assumption are only necessary and sufficient if applied either to a unitary state or a federal economy in which the provincial sector tax finances changes to its budget position.

In fact, when agents perceive bonds to be part of their net wealth, both stability conditions and actual steady state levels of the debt-to-GDP ratio are changed. In the case of the unitary state, because of the ambiguous relationship between bonds and inflation, the direction and magnitude of these change are uncertain. In the federal economy

case, choice of financing regimes alters the results. In any case, simulation exercises in the following section evaluate quantitatively stability conditions and equilibrium debt-to-GDP values under alternative assumptions about the model and associated parameters.

4. Simulation Models: Results

The existence of a bias in the equal equation first difference analysis has been identified but its direction and magnitude is ambiguous. This section presents the results of simulation exercises aimed at shedding light on this issue. In particular, the impact of a tax shock on equilibrium values of the debt-to-GDP ratio, interest rates and inflation is calculated under varying assumptions about the parameters and the value of β for both the unitary state and federal state cases. The analysis is based on the empirical versions of the models specified in the previous section and presented in Appendix A.

As is evident in many simulation studies, the choice of specific parameter values is important. In this study, the parameters in the consumption and money demand functions are of particular interest. Table 1 details values of the coefficients for the two cases considered. While many other values are possible, for the illustrative purposes of this paper, the analysis is limited to two cases.

Table 1

Alternative Values of the Parameters
(elasticities unless otherwise indicated)

	Case 1	Case 2
C_1*	0.700	0.700
C_2	-0.100	-0.100
C_3*	0.045	0.000
L_1	0.900	0.900
L_2	-0.500	-0.500
L_3	0.200	0.000

Note:
*Marginal propensities

The two cases differ only in the values assumed for C_3 and L_3. The marginal propensity to consume out of disposable income, C_1, is slightly higher than the 0.6 value assumed by Duguay and Rabeau (1987) but lower than the 0.9 value assumed by Scarth (1987). The elasticity of consumption with respect to the expected real interest rate determines C_2, and is just above the average parameter value used by Nguyen and Turnovksy (1979) of -0.18. In Case 1, the average propensity to consume out of wealth, determining C_3, is assumed to be the same as in Duguay and Rabeau (1987). Case 2 follows Scarth (1987) and sets both C_3 and L_3 equal to zero. As later discussed, this has a significant impact on the results. The elasticity of money demand with respect to income, determining L_1, falls within the range of values employed by Nguyen and Turnovsky (1979). Since Laidler (1977) found that, in general, empirical studies found the elasticity of money demand with respect to the interest rate equal to -0.5, this value is used in determining L_2. In Case 1, the elasticity of money demand with respect to wealth is the same as in Kneebone (1988).

The initial values of the system are chosen from the National Accounts and Financial Flows Accounts to approximately reflect the Canadian situation in 1987.[10] Given that the model presents fewer details than the National Accounts and represents a closed economy, it is not possible to calibrate exact 1987 values. In this respect, the results are only a stylized characterization of the Canadian economy. The interest rate is initialized at 7 per cent, the exogenously determined contemporaneous rate of growth in real output is set at 3.5 per cent, and inflation is set at 4 per cent.[11]

Recall that when $\beta = 0$, the model dichotomizes and becomes analogous to the single equation first difference framework. As a result, the following examination will compare the results to the Ricardian case where $\beta = 0$.

(i) Unitary State

Utilizing a discrete time version of the unitary state model developed in the previous section, the impact of a 1 per cent decline in federal taxes financed by bonds is examined. Since the single level of

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10. It is important to note that the debt levels used in the calibration are from the Financial Flows Accounts.

11. These values are chosen for simulation purposes to broadly reflect their historical averages and to ensure stability of the debt-to-GDP ratio in the Ricardian case. Averages for the 1971 to 1987 period indicate that the contemporaneous rate of real growth in GDP was 4 per cent, the effective interest rate on net debt was 7.1 per cent and the contemporaneous rate of GDP inflation was 6.5 per cent. Given that inflation was relatively high over this period by historical standards, a lower rate was chosen for simulation purposes.

government is assumed to be federal to facilitate comparisons drawn in the latter part of this section, minor alterations to the GDP identity are required for calibration purposes and are presented in Appendix A. Specifically, given an initial equilibrium position of 7 per cent interest rates, 4 per cent inflation and a 40 per cent debt-to-GDP ratio, the magnitude of change in these equilibrium values is calculated.

In Case 1, the impact on the equilibrium debt-to-GDP ratio of increasing the value of β is dramatic. When $\beta = 0$, which is equivalent to the first difference analysis, the debt-to-GDP ratio increases over 8 percentage points to 48.1 per cent after the shock. Recalling equation (3), the equilibrium debt-to-GDP ratio in the single equation first difference framework, verifies this result.[12] As illustrated by Chart 1, the equilibrium debt-to-GDP ratio is progressively higher at higher values of β. In particular, when agents perceive 60 per cent of government bonds to be part of net wealth ($\beta = 0.6$), the equilibrium debt-to-GDP ratio is 2.3 percentage points higher than in the pure Ricardian case. If agents perceive all government bonds to be part of their net wealth, the equilibrium debt-to-GDP ratio is over 4 percentage points higher than in the single equation-first difference case, stabilizing at 52.2 per cent.

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12. Equation (3) specifies that the steady-state debt-to-GDP ratio in the single equation first difference framework is defined as follows:

$$b(e) = \frac{pd}{n - [r(1 - t) - p]}$$

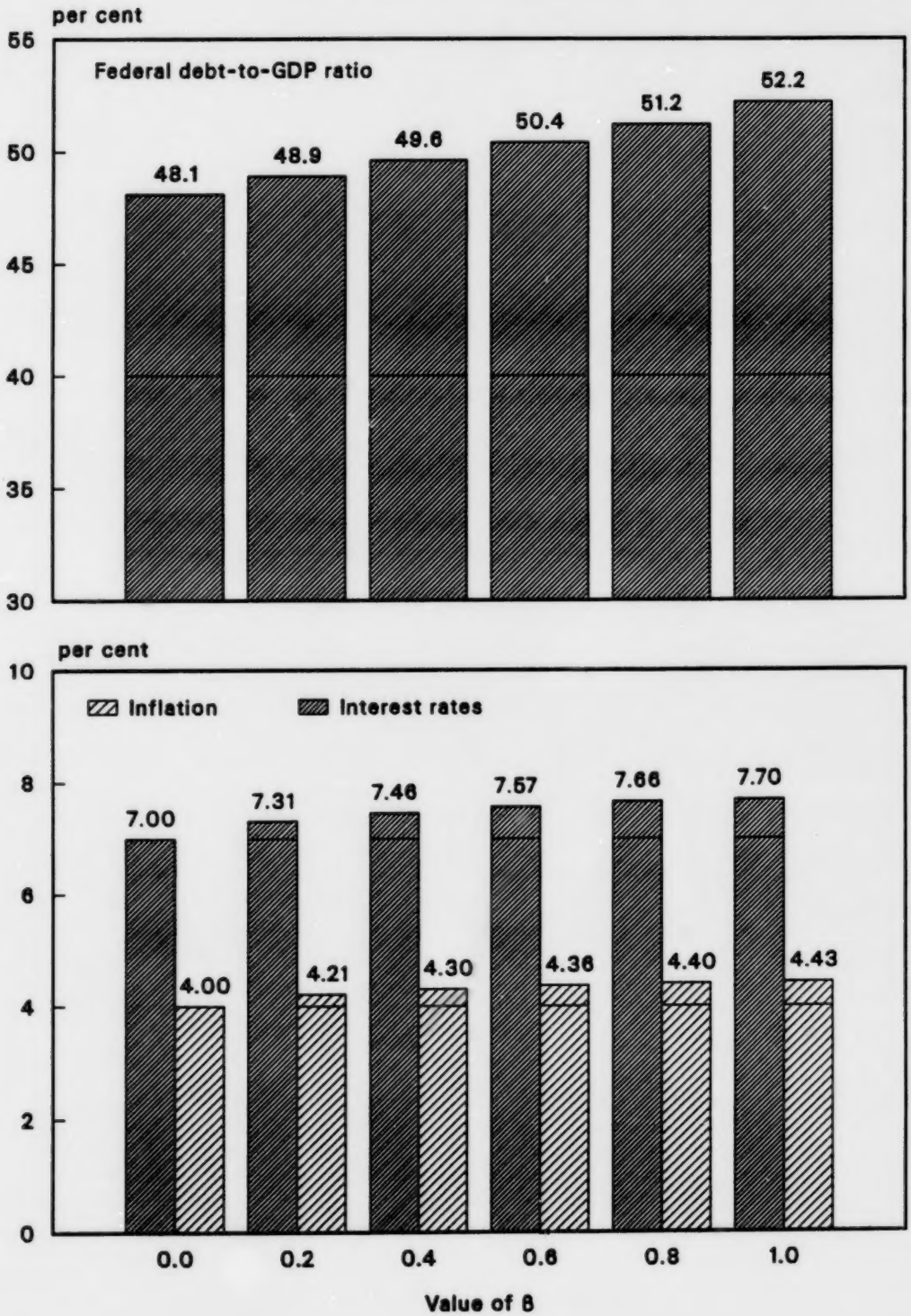
The explanation supporting these results is best revealed through consideration of the stability conditions. Specifically, when $\beta = 0$ (the single equation-first difference case) the growth greater than the real after tax interest rate effect is enough to ensure stability. Therefore, after the tax shock, this effect causes the debt-to-GDP ratio to stabilize at a new (higher) level. However, if β is not equal to zero, the effects of an increase in bonds on both interest rates and inflation become important factors in the stability of the system and on its steady-state values. For example, when $\beta = 0.2$ the increase in interest rates to 7.3 per cent, through its effect on debt interest payments, has a negative impact on stability. In addition, the increase in inflation to 4.2 per cent, through the inflation tax, has a positive impact on stability. In this case, the negative impact on stability from an increase in interest rates dominates and decreases the degree to which the stability conditions are satisfied relative to the $\beta = 0$ case. As a result, the debt-to-GDP ratio stabilizes at the slightly higher level of 48.9 per cent. Thereafter, the negative impact of interest rates on stability always outweighs the positive impact from higher inflation and the debt-to-GDP ratio stabilizes at much higher levels.

When the dependence of consumption and money demand on wealth is eliminated, as in Case 2, the equilibrium debt-to-GDP ratio increases as β increases, but for different reasons. In fact, as illustrated in Chart 2, although interest rates remain at 7 per cent after the shock, the debt-to-GDP ratio stabilizes at successively higher levels as the

Chart 1

EFFECTS OF A ONE PER CENT DECREASE IN TAXES FINANCED BY BONDS

Unitary State: Case 1



assumption of Ricardian equivalence is relaxed. For example, if agents perceive 80 per cent of government bonds to be part of net wealth the debt-to-GDP ratio stabilizes at 50.2 per cent, over 2 percentage points higher than in the single equation first difference framework.

The explanation for this result can, again, be thought of in terms of the impact of an increase in government bonds on interest rates and inflation. In this case, interest rates do not change because, when the elasticity of money demand with respect to wealth is assumed to be zero ($L_3 = 0$), the positive effect of an increase in bonds on wealth does not influence money demand. Recalling equation (16), and noting that the partial derivative of r with respect to b is zero if L_3 is equal to zero verifies this point.[13] However, in this case, an increase in government bonds results in a small decline in inflation. Thinking in terms of the IS-LM framework, increasing government bonds results in a rightward shift of the IS curve through the transfer effect only (since the direct wealth effect on consumption is zero). This results in an excess demand situation in which, since nominal interest rates are fixed, real interest rates must adjust. As a result, there is a reduction in inflation and hence, an increase in the real interest rate. This, in turn, has a negative impact on stability conditions and the equilibrium debt-to-GDP ratio.

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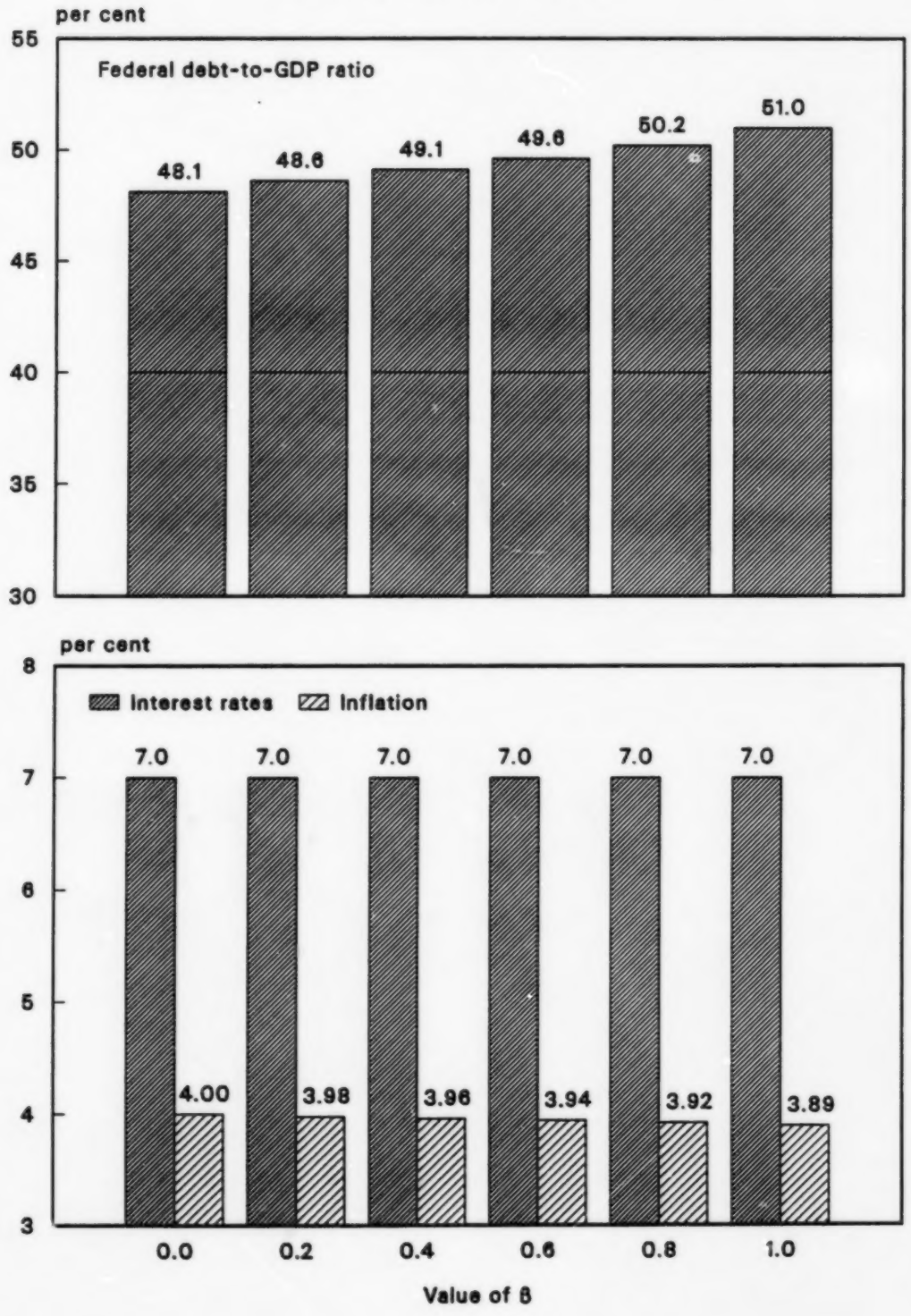
13. Equation (16) specifies that the response of interest rates to a change in government bonds is of the form:

$$\frac{\partial r}{\partial b^f} = -\frac{\beta L_3}{L_2} \geq 0$$

Chart 2

EFFECTS OF A ONE PER CENT DECREASE IN TAXES FINANCED BY BONDS

Unitary State: Case 2



(ii) Federal Economy

A discrete time version of the federal economy model developed in the theoretical section is employed to examine the impact of a 1 per cent decline in federal taxes financed by bonds in the case where the provincial sector bond finances changes to their budget positions.[14] Given the same equilibrium position as in the unitary state case -- excepting debt-to-GDP ratios of the federal and provincial levels which are 40 per cent and 10 per cent respectively -- the change in the equilibrium values of interest rates, inflation and debt-to-GDP ratios is calculated.

In Case 1, as illustrated by Chart 3, the shock has an increasingly large impact as the value assumed for β increases. When $\beta = 0$, the pure Ricardian case, the federal equilibrium debt-to-GDP ratio increases over 12 percentage points to 52.4 per cent while the provincial equilibrium debt-to-GDP ratio declines by over 5 percentage points to 4.4 per cent. This result exemplifies the link between the two government budget constraints. Specifically, a decrease in federal tax revenue financed by federal government bonds causes the federal

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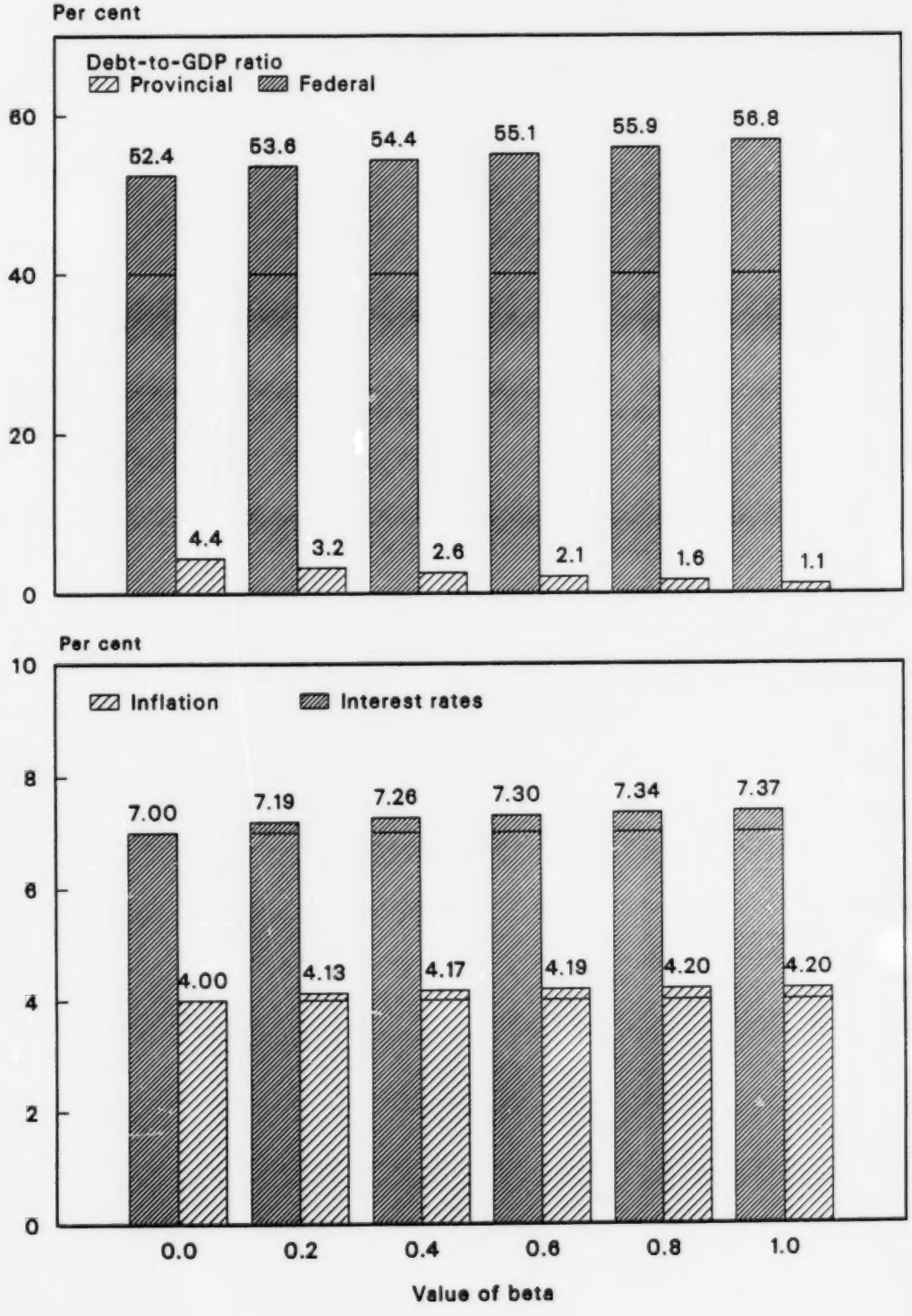
14. In practice there is a strong interrelationship between the federal and provincial tax structure, principally in the case of the corporate and personal income tax. In this exercise we assume that the shock at the federal level does not affect the provincial tax structure. Assuming a 1 per cent decline in only personal and corporate income tax at the federal level would result in a reduction of no more than 0.1 per cent in provincial tax. The interest in such a shock is then marginal.

Chart 3

EFFECTS OF A ONE PER CENT DECREASE IN FEDERAL TAXES: FEDERAL AND PROVINCIAL BOND FINANCING

Federal State: Case 1



debt-to-GDP ratio to rise. Provincial revenue, from tax on federal bond interest income, will rise. At the provincial level, this implies a smaller need to bond finance, and hence a lower debt-to-GDP ratio in equilibrium. When agents perceive some fraction of government bonds to be part of net wealth this result is magnified. For example, when $\beta = 1$ the federal equilibrium debt-to-GDP ratio rises to 56.8 per cent, over 4 percentage points higher than in the pure Ricardian case. Similarly, at this level of β, the provincial equilibrium debt-to-GDP ratio falls to 1.1 per cent, over 3 percentage points lower than in the pure Ricardian case.

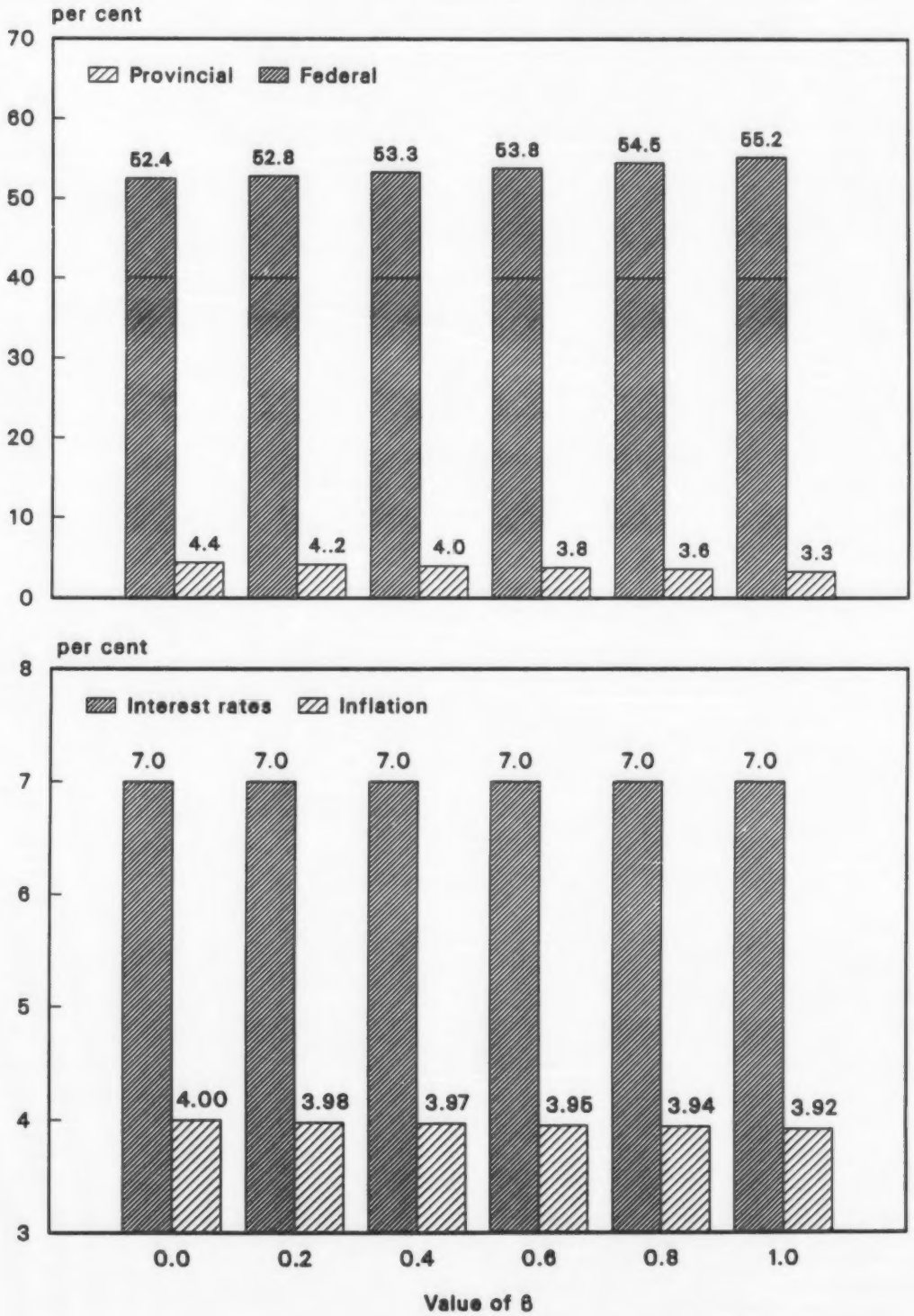
Examination of the change in interest rates and inflation induced by an increase in federal government bonds reveals similar results to the unitary state case. Both interest rates and inflation stabilize at higher levels as the assumption of Ricardian equivalence is relaxed such that, at every value of β assumed, the federal equilibrium debt-to-GDP ratio rises.

In Case 2, as illustrated by Chart 4, when C_3 and L_3 are assumed to be equal to zero the impact of the shock on both the federal and provincial debt-to-GDP ratios is the same as in the previous case. However, for the same reasons as in the unitary state model, the difference between equilibrium debt-to-GDP ratios in the Ricardian and non-Ricardian scenarios stems solely from a decrease in the inflation rate. It is interesting to note that, in the federal state model, the

Chart 4

EFFECTS OF A ONE PER CENT DECREASE IN FEDERAL TAXES FEDERAL AND PROVINCIAL BOND FINANCING

Federal State: Case 2



decrease in the inflation tax affects both budget constraints such that the provincial equilibrium debt-to-GDP ratio declines less in Case 2 than in Case 1 (where inflation tax actually increases) as the assumption of Ricardian equivalence is relaxed.

Theoretical results in Section 3 indicate that the presence of a second budget constraint renders the stability conditions more difficult to achieve, regardless of the assumption of Ricardian equivalence. Comparing the results of the unitary state model to the federal state model reveals that a 1 per cent decline in tax increases the equilibrium debt-to-GDP ratio more when the provincial government sector budget constraint is added at all levels of β. For example, when $\beta = 1.0$ (Case 1) the equilibrium debt-to-GDP ratio at the federal level increases 12.2 percentage points in the unitary state model compared to 16.8 percentage points in the federal state model. This results from the interaction between the two budget constraints and illustrates the serious risk of underestimating equilibrium debt-to-GDP ratios at the federal level when the provincial government budget constraint is ignored.

In both the unitary state and federal state models it is apparent that the impact of the shock on equilibrium debt-to-GDP ratios is directly proportional to the value of β. For low values of β (β=0.2) the difference between the single equation first difference analysis and the full model is small. However, the discrepancy between the single equation first difference approach and the IS-LM framework becomes more

significant as the value of β increases. This indicates that, for the two sets of coefficients considered, the net impact of an increase in government bonds on interest rates and inflation worsens the stability conditions (and hence steady state debt-to-GDP values).

5. Conclusions

Current theory indicates that the robustness of the results of the single equation first difference analysis relies on a very restrictive macroeconomic context. Specifically, the assumption of parameter constancy is only valid in a Ricardian world. In fact, stability conditions derived under this assumption are only necessary and sufficient if applied to a unitary state or a federal economy in which the provincial sector tax finances changes to their budget positions. In the unitary state model, relaxing the Ricardian equivalence assumption adds two new elements to the stability conditions: the impact of a change in interest rates on interest paid on government debt, and the change in inflation tax from a change in debt. In the federal economy model, when the provincial sector is assumed to bond finance, the standard stability conditions are neither necessary nor sufficient to achieve a full equilibrium regardless of the Ricardian equivalence assumption.

Given that a bias from a single equation first difference framework does exist in a non-Ricardian, federal economy, simulation exercises in the previous section are revealing. Specifically, in both

the unitary and the federal state models, it is apparent that the impact of the tax shock on equilibrium debt-to-GDP ratios is more severe as the Ricardian equivalence assumption is relaxed. In the unitary state model, the discrepancy between the single equation first difference framework and the full model equilibrium debt-to-GDP ratio can be as large as 4 percentage points (Case 1) given a 1 per cent decrease in tax revenue. Inclusion of a provincial sector in the federal model reveals a link between the two level of governments' budget constraints. In particular, the federal tax shock results in an improvement in the provincial debt-to-GDP ratio at the expense of the federal government. As well, simulations reveal significant underestimation of steady-state debt-to-GDP ratios at the federal level when the provincial government sector budget constraint is ignored.

The analysis provided in this paper could usefully be extended in several ways. First, given the open nature of the Canadian economy, there may be positive returns to extending the analysis to an open economy model. Second, since growth is a key variable in the entire analysis, it would be interesting define it as an endogenous variable and examine the link between government debt and growth. With the first suggestion in mind, the development of an open economy version the unitary state model is currently underway.

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APPENDIX A

Appendix A

Simulation model specification and calibration procedure.

In order to perform numerical simulations, the models presented in Section 3 have to be slightly altered. First, the model must be rewritten in discrete time. Second, the consumption and money demand functions must be given the necessary properties to guarantee the realization of a numerical steady-state. Finally, consistent values for the coefficients must be calibrated. These issues will be examined successively.

Specification of a discrete time model.

For simulation purposes the models presented in section 3 must be translated into a discrete time formulation. While building a theoretical model, it is not generally required to introduce a specific notion of time periods per se. Multipliers and dynamic results can be obtained based on differential calculus which assumes that events occur in a continuum, not in discrete time periods. The task of translating the continuous time models outlined in Section 3 into discrete time versions is straightforward and requires only a few considerations. First, to account for the fact that interest payments received at the end of a period are a function of the outstanding stock of debt at the beginning of the period, the stock of government debt is lagged one period in the money demand, wealth and consumption functions in the unitary and federal economy versions of the model. Second, disposable

income is derived in a discrete time framework using the same definition outlined previously and differs slightly from its continuous time counterpart. Finally, in the government budget constraints, nominal government bonds at time t-1 are deflated by prices at time t. Given that there is only one price in the model, expressing the government budget constraint in real terms implies dividing both sides of the relationship by price at t. The unitary state and federal state models are outlined in Figure 1 and Figure 2 respectively.

Consumption and money demand functions.

In order to achieve a steady-state, consumption and money demand functions must be homogeneous of degree one with respect to the flow variables. For example, if income, wealth and GDP are doubled, these functions must be defined in a way that demand for goods and money will also double. If this property does not exist, nominal interest rates and inflation will have to adjust endlessly and steady-state is not achievable.

Based on simple linear functions incorporating a constant term as used in the model, some adjustments have to be made in order to achieve this result. The constant term and the coefficient on the real or nominal interest rate have to be multiplied by the ratio of the current GDP to the base year GDP (GDP/(t)/GDP(0)). It is straightforward to verify that this correction produces the desired homogeneity properties.

Calibration of the model.

Calibration of the model involves mainly three steps: calculating the coefficients in the model given the assumed elasticities, determination of a steady-state and choosing initial values for the variables.

a) Coefficients in the model.

Calculation of values for the coefficients given the assumed elasticities involves simple mathematics. For example, since

$$C/(r-p) = C_2$$

and the elasticity of consumption with respect to expected real interest rates is

$$\partial C/\partial(r-p)*(r-p)/C$$

and assuming an elasticity of consumption with respect to the expected real interest rate of -0.1, the implied value for C_2 is:

$$-0.1*C/(r-p)$$

The same procedure is employed to find L_1, L_2 and L_3. Once all coefficients are determined, the constants C_0 and L_0 are determined residually.

b) Determination of a steady-state.

Provided that the model is stable and given initial values for the parameters and lagged endogenous variables it should be possible to generate a steady-state debt-to-GDP ratio in the control simulation. However, considering that comparative dynamic results are those of interest in this paper, steady-states are imposed in the control simulation. Specifically, the debt-to-GDP ratio remains stable at its initial value in the control simulation by adjusting the average tax rates in the budget constraint(s). Table 1 outlines the resulting values.

Table 1
Implicit Tax Rates
(per cent)

	Unitary State Model	Federal State Model
$TAXA(t^{y})$	38.71	n.a.
$TAXFA(t^{fy})$	n.a.	16.39
$TAXPA(t^{py})$	n.a.	21.97
$TAXB(t^{r})$	19.24	n.a.
$TAXFB(t^{fr})$	n.a.	20.00
$TAXPB(t^{pr})$	n.a.	10.00

c) Initial Values.

The initial values of the system, as shown in Table 2, are taken from the National Accounts to approximately reflect the Canadian situation in 1987. For reasons outlined already in the body of the

paper, this is only meant to represent a stylized characterization of the Canadian economy. To facilitate comparison of the two simulation models, a new variable denoted AC is introduced into the GDP identity in the unitary state model. Treated as autonomous consumption, AC ensures that GDP is equivalent in both models.

Table 2
Initial Values: 1987
($ billion unless indicated otherwise)

	Unitary State Model	Federal State Model
Y	550.0	550.0
YPOT	550.0	550.0
C	430.0	430.0
GF	25.0	25.0
GP	n.a.	95.0
AC	95.0	n.a.
BF	215.0	215.0
BP	n.a	60.0
M	55.0	55.0
TRANSF	70.0	70.0
TRANSP	n.a	28.0
R (%)	7.0	7.0
INF (%)	4.0	4.0
GYPOT (%)	3.5	3.5
P (index)	1.0	1.0

Figure 1

```
MODEL: MARIOPT1

SYMBOL DECLARATIONS

ENDOGENOUS:
   AC  BF  C  DEBTRF  GF  INF  M  P  PI  R  TRANSF  W  Y  YD  YPOT

EXOGENOUS:
   TAXFA  TAXFB

COEFFICIENT:
   BETA  C0  C1  C2  C3  GAMMA  GYPOT  L0  L1  L2  L3

EQUATIONS

     1:        Y = C+GF+AC
     2:        C = C0*Y/550.+C1*YD+C2*Y/550.*(R-PI)+C3*W(-1)
     3:        W = M+BETA*BF/P
     4:        GF = 0.045454*Y
     5:        AC = 0.172727*Y
     6:        TRANSF = 0.127273*Y
     7:        YD = Y+R*BF(-1)/P-TAXFA*Y-TAXFB*R*BF(-1)/P-(P-P(-1))/P*W(-1)-(1-BETA)*((BF-BF(-1))/P)+TRANSF
     8:        M = L0*Y/550.+L1*Y+L2*Y/550.*R+L3*W(-1)
     9:        BF-BF(-1) = P*GF+P*TRANSF-TAXFA*P*Y+R*(1-TAXFB)*BF(-1)
    10:        DEBTRF = BF/(P*Y)
    11:        INF = GAMMA*(Y-YPOT)+PI
    12:        P = (1+INF)*P(-1)
    13:        PI = INF
    14:        M = 0.1*Y
    15:        YPOT = YPOT(-1)*(1+GYPOT)
```

MODEL: MARIOPT

SYMBOL DECLARATIONS

ENDOGENOUS:
BF BP C DEBTRF DEBTRP GF GP INF M P PI R TRANSF TRANSPR W Y YD YPOT

EXOGENOUS:
TAXFA TAXFB TAXPA TAXPB

COEFFICIENT:
BETA C0 C1 C2 C3 GAMMA GYPOT L0 L1 L2 L3

EQUATIONS

1: Y = C+GF+GP

2: C = C0*Y/550.+C1*YD+C2*Y/550.*(R-PI)+C3*W(-1)

3: W = M+BETA*(BF/P+BP/P)

4: GF = 0.045454*Y

5: GP = 0.172727*Y

6: TRANSF = 0.127273*Y

7: TRANSPR = 0.050909*Y

8: YD = Y+R*(BF(-1)/P+BP(-1)/P)-(TAXFA+TAXPA)*Y-(TAXFB+TAXPB)*R*(BF(-1)/P+BP(-1)/P)-(P-P(-1))/P*W(-1)-(1-BETA)*((BF-BF(-1))/P)-(1-BETA)*((BP-BP(-1))/P)+TRANSF+TRANSPR

9: M = L0*Y/550.+L1*Y+L2*Y/550.*R+L3*W(-1)

10: BF-BF(-1) = P*GF+P*TRANSF-TAXFA*P*Y+R*(1-TAXFB)*BF(-1)-TAXFB*R*BP(-1)

11: BP-BP(-1) = P*GP+P*TRANSPR-TAXPA*P*Y+R*(1-TAXPB)*BP(-1)-TAXPB*R*BF(-1)

12: DEBTRF = BF/(P*Y)

13: DEBTRP = BP/(P*Y)

14: P = (1+INF)*P(-1)

15: INF = GAMMA*(Y-YPOT)+PI

16: PI = INF

17: M = 0.1*Y

18: YPOT = YPOT(-1)*(1+GYPOT)